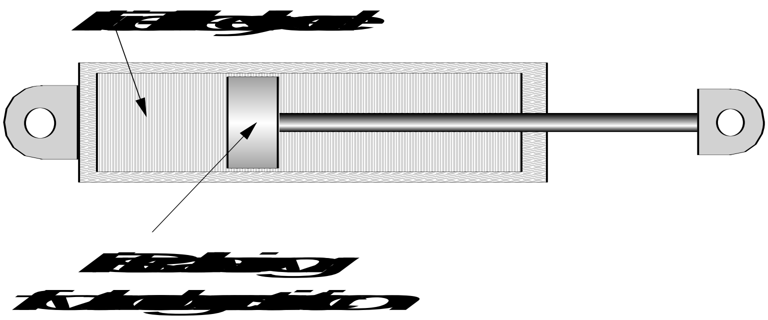
FORCED VIBRATION OF SINGLE DEGREE-OF-FREEDOM SYSTEMS
MODULE: MMME2046 DYNAMICS & CONTROL

So far, we have looked at free vibration of systems and at their natural frequencies and mode shapes. This section will analyse the response of single degree-of-freedom systems to external excitation. This takes the form either of applied forces and/or moments or of imposed displacement on part of the system.

We will also introduce the effects of damping. ***Damping*** is a phenomenon characterising the dissipation of energy in a structure. If there was no damping and a structure was set vibrating and then left, the mathematics would suggest that it would carry on vibrating forever. This, of course, is impossible and the structure would stop. This is because all real structures dissipate energy. There are many ways this can take place and mechanisms include hysteresis effects in the material, friction between parts, aerodynamic interaction with the surrounding fluid and noise radiated from the surfaces. Most real damping mechanisms are difficult to handle mathematically and we will consider one theoretical damping model, called ***viscous damping*** and will only consider discrete dampers. These can be pictured as a piston-cylinder device in which a viscous fluid is displaced from one side of the piston to the other through a constriction such as an orifice. Vehicle "shock absorbers" (dampers) are normally based on this idea.



In the viscous damping model, we assume that the force in the damper is proportional to the relative velocity between the ends and acts to oppose the imposed motion. The constant of proportionality is called the damping coefficient (normally given the symbol c) and has units of N/(m/s) [or Ns/m].

Hence the force opposing the motion is

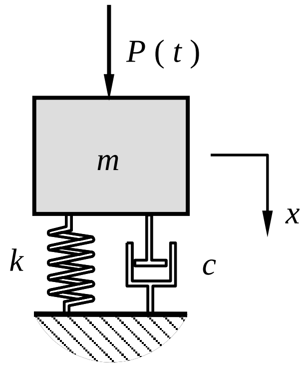
Note that dampers do not impose any stiffness on the structure; they only transmit a force if there is relative motion between the ends. If there is no motion, there is no force.

In most engineering structures, the level of damping is low. As a result, any discrepancies between the assumed viscous damping model and the true damping mechanism are generally small, so that the error introduced by our mathematical model is also small.

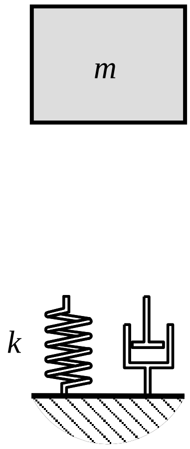
In the examples that follow, the steps leading to the equation of motion are the same as before with the addition of any damping forces and any external excitation. We will then look at the solution to the equation of motion for a few types of excitation.

Example 1 Mass-Spring-Damper System

STEP 1: Dynamic mass-spring model



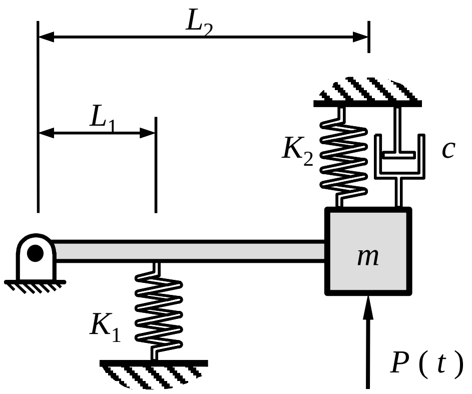
STEP 2: Free Body Diagram



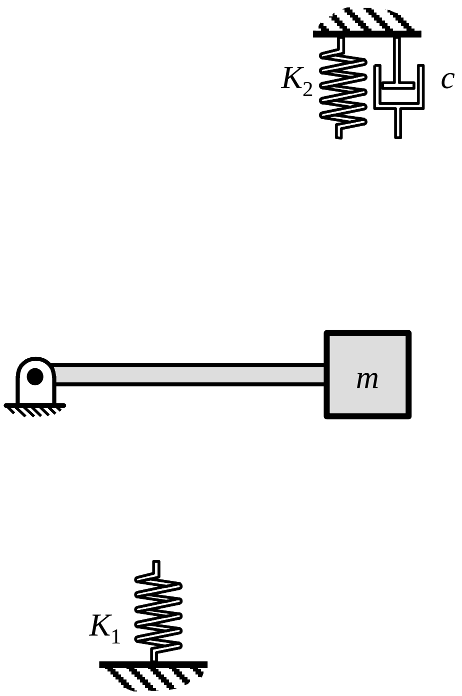
STEP 3: Equation of motion

Example 2 Rocker system

STEP 1: Dynamic mass-spring model



STEP 2: Free Body Diagram
(for small displacements)



STEP 3: Equation of motion

Example 3 Single-axle caravan

We will make the following simplifying assumptions.

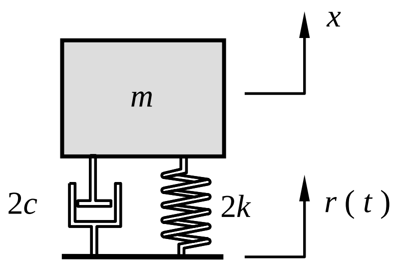
1 The tyres are very stiff compared to the suspension springs (typically they are about 10 times stiffer).

2 The tyres do not lose contact with the road. Taken with Point 1 this means that vertical motion of the axle will follow the road profile exactly.

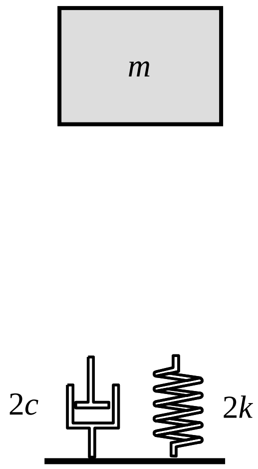
3 The caravan body behaves as a rigid mass.

4 Only vertical motion of the body is considered; pitching and rolling are ignored.

STEP 1: Dynamic mass-spring model



STEP 2: Free Body Diagram



The free-body diagram is a "snapshot" of the system when **all motions** (both displacements and velocities) are **positive**. It shows the **positive directions** of the forces that the springs and dampers exert on the mass. The expression for the force in the spring is given by ***Spring force = Stiffness* x *Change of length***, so we ask:

- What is the **change of length** of the spring, noting that both ends of the spring move in this problem?
- Is the spring in **tension** or **compression**?

For the damper, ***Damper force = Damping coefficient* x *Relative velocity***, so we ask:

- What is the **relative velocity** between the ends of the damper, noting again that both ends are moving?
- Is the damper being **extended** or **compressed**?

STEP 3: Equation of Motion

This is an example where the excitation is in the form of a prescribed **displacement** instead of a force. For any given vehicle speed, the shape of the road profile, $r(t)$, can be measured as an explicit function of time, so that we know exactly what the displacement of the axle will be. Differentiating the displacement give $\dot{r}(t)$, again as an explicit function of time. As a result, the excitation term on the RHS of the equation of motion is completely defined.

Warning: A common error made by students is to treat the displacement of the axle as if it was a "force" applied directly to the body. This is wrong. The forces experienced by the body due to the movement of the axle are applied through the springs and dampers and depend on the change of length of the spring and the relative velocity across the damper, respectively.

Summary so far

The equations of motion for the three examples are as follows.

Mass-spring-damper system $\quad m\ddot{x} + c\dot{x} + kx = P(t)$

Rocker system $\quad mL_2^2\ddot{\theta} + cL_2^2\dot{\theta} + \left(K_1L_1^2 + K_2L_2^2\right)\theta = L_2P(t)$

Single-axle caravan $\quad m\ddot{x} + 2c\dot{x} + 2kx = 2c.\dot{r}(t) + 2k.r(t)$

While each of the equations is different in detail, you will see that they all share a common mathematical form, that of a **second-order ordinary differential equation with constant coefficients.** All linear, single-degree-of-freedom systems have this form, which can be written generically as:

$$M\ddot{z} + C\dot{z} + Kz = F(t) \tag{1}$$

in which
z is the response coordinate
M is the coefficient of $\ddot{z}$
C is the coefficient of $\dot{z}$
K is the coefficient of z and
$F(t)$ is the excitation function; independent of z

The exact form of the 3 coefficients and of the excitation function will depend uniquely on the system being analysed.

Remember that **every term in the expressions for the coefficients M, C and K must be positive**[1] and that **any negative sign means that *your equation is definitely wrong*.**

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1 Depending on the positive directions chosen for the response, the excitation can be either positive or negative.

There is no point in proceeding until you have found the error. If you find a negative sign, go back and first check your free-body diagram and then check if you have applied Newton's 2nd Law correctly (in particular, checking that you have resolved the forces/moments in the direction of the motion coordinate).

Solutions to the equation of motion

The subsequent mathematical manipulation required to solve the equation of motion depends on the nature of the excitation function and on the amount of damping in the system. A total of 6 cases will be considered, divided into 3 sections. You must learn to recognise the various cases so that you can apply the appropriate solution procedure.

A: "FREE" VIBRATION

Case (i) Zero damping
Case (ii) High damping
Case (iii) "Critical" damping
Case (iv) Low damping

B: RESPONSE TO HARMONIC (SINUSOIDAL) EXCITATION

C: RESPONSE TO ARBITRARY PERIODIC EXCITATION

Section A is covered below and there are separate handouts for Sections B and C.

Section A: "FREE" VIBRATION

If no externally applied forces/moments act on a structure, it can vibrate freely; hence the term "free" vibration. We have previously used this situation to find the natural frequency of undamped systems. As mentioned above, the presence of damping means that any vibration will stop sooner or later and consequently the term "transient response" is often used in place of free vibration.

For $F(t) = 0$, we can use a solution in the form, $z(t) = A\mathrm{e}^{\lambda t}$

Substituting into the equation of motion gives,

$$M\,.\,\lambda^2 A\mathrm{e}^{\lambda t} + C\,.\,\lambda A\mathrm{e}^{\lambda t} + K\,.\,A\mathrm{e}^{\lambda t} = 0$$

For a non-trivial solution, $M\lambda^2 + C\lambda + K = 0$

so that

$$\lambda_{1,2} = \frac{-C \pm \sqrt{C^2 - 4KM}}{2M} \qquad (2)$$

The complete solution is then

$$z(t) = A_1\,\mathrm{e}^{\lambda_1 t} + A_2\,\mathrm{e}^{\lambda_2 t} \qquad (3)$$

The integration constants, A_1 and A_2, are found from the "initial conditions" specified in the problem. These will normally be the displacement and velocity at t = 0. The values may be stated, or you may have to work them out from information in the problem.

It can be seen from equation (2) that the roots $\lambda_{1,2}$ can be either real or complex, depending on the amount of damping present. While equation (3) gives a mathematically correct description of the response, other forms relate more closely to engineering intuition and are thus easier to interpret. Four cases with different damping levels are considered.

CASE (i) ZERO DAMPING

For zero damping, we know that the system will oscillate with simple harmonic motion, although the sinusoidal waveform is not obvious from equation (3).

For C = 0, $$\lambda_{1,2} = \pm \frac{\sqrt{-4KM}}{2M} = \pm i\sqrt{\frac{K}{M}}$$

The term $\sqrt{\frac{K}{M}}$ is called the "***undamped natural frequency***" and is given the symbol ω_n. Previously, we used the term "natural frequency" for this. As we shall see later, a damped system will have a frequency at which it will vibrate freely, so the words "undamped" or "damped" are used to distinguish between the two.

Returning to the general case, equation (3) becomes

$$z(t) = A_1\, e^{i\omega_n t} + A_2\, e^{-i\omega_n t}$$

This still does not look much like a sinusoidal waveform. However, by using the complex number identities,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

and the fact that A_1 and A_2 are a complex conjugate pair, you should be able to show that

$$z(t) = B\cos\omega_n t + C\sin\omega_n t \qquad (4)$$

Try this as an exercise.

Once you have recognised that a problem is one of free vibration with zero damping, it will normally be most convenient (and will certainly give a solution which is easiest to interpret) if equation (4) is used rather than equation (3). Again, the constants B and C are found from the initial conditions of the problem.

<u>Exercise</u> If an undamped structure is given an initial displacement Z_0 and then released from rest, show that the subsequent response is given by

$$z(t) = Z_O\cos\omega_n t$$

CASE (ii) HIGH DAMPING

If the damping level is high so that $C > 4KM$, the two roots, $\lambda_{1,2}$, are both REAL and NEGATIVE. The response, as given by equation (3), is the sum of 2 decaying exponential functions. The constants A_1 and A_2 are found from the initial conditions as usual.

Exercise If a heavily damped structure is given an initial displacement Z_O and then released from rest, show that the constants of integration are

$$A_1 = \frac{Z_O \lambda_2}{\lambda_2 - \lambda_1} \qquad A_2 = \frac{-Z_O \lambda_1}{\lambda_2 - \lambda_1}$$

Sketch the graph of $z(t)$ against time.

CASE (iii) "CRITICAL" DAMPING

A special case for the roots of equation (2) occurs if $C^2 = 4KM$. This value of damping is known as "critical damping", which is thus given by

$$C_{crit} = 2\sqrt{KM} \tag{5}$$

This is an important relationship, but again, you must remember that K and M are the **coefficients** from the equation of motion and may be related to individual springs and masses in the system in a complicated way.[2]

From equation (2) it will be seen that $\lambda_1 = \lambda_2 = -\dfrac{C_{crit}}{2M} \equiv -\omega_n$

In this situation, in order to maintain distinct parts to the solution, the response is given by

$$z(t) = A_1 e^{-\omega_n t} + A_2\, t\, e^{-\omega_n t} \tag{6}$$

Note the "t" in the second term.

Exercise A critically damped structure is subjected to an impulse such that it acquires an instantaneous initial velocity, V_O, while the displacement remains zero. Show that the subsequent displacement is given by,

$$z(t) = V_O\, t\, e^{-\omega_n t}$$

2 For example, the critical damping expression for the rocker system is

$$2\sqrt{m L_2^2 \left(K_1 L_1^2 + K_2 L_2^2\right)}$$

CASE (iv) LIGHT DAMPING

The vast majority of engineering structures possess damping levels much less than critical. From common experience, we know that a structure left to vibrate freely will come to rest eventually. The light damping case is thus the norm. Mathematically, this corresponds to the case where $C^2 < 4KM$ and the roots of equation (2) are a complex conjugate pair, viz.

$$\lambda_{1,2} = -\frac{C}{2M} \pm i\frac{\sqrt{4KM - C^2}}{2M} \tag{7}$$

It is convenient to introduce the ***damping ratio***, $\gamma = \dfrac{C}{\text{critical damping}} = \dfrac{C}{2\sqrt{KM}}$

This formula is on the formula sheet, but please remember that M, C and K are **coefficients from the equation of motion**.

Using also the expression for the undamped natural frequency, ω_n, equation (7) becomes

$$\lambda_{1,2} = -\gamma\omega_n \pm i\omega_n\sqrt{1-\gamma^2} \tag{8}$$

Substitution into equation (3) gives a mathematically correct (but not very convenient) solution.

$$z(t) = A_1 e^{\left(-\gamma\omega_n + i\omega_n\sqrt{1-\gamma^2}\right)t} + A_2 e^{\left(-\gamma\omega_n - i\omega_n\sqrt{1-\gamma^2}\right)t} \tag{9}$$

Again, making use of the complex exponential identities and the fact that A_1 and A_2 are a complex conjugate pair, equation (9) can be re-written as

$$z(t) = e^{-\gamma\omega_n t}\left[B_1\cos\left(\omega_n\sqrt{1-\gamma^2}\,t\right) + B_2\sin\left(\omega_n\sqrt{1-\gamma^2}\,t\right)\right] \tag{10}$$

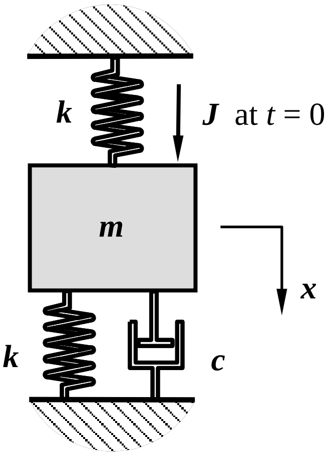
Equation (10) matches our intuition since it describes a sinusoidal waveform (indicated by the terms in the square brackets) with an exponentially decaying term at the front that will cause the amplitude of the sinusoid to decrease. Note that the constants B_1 and B_2 are both real. As an alternative, equation (10) can be re-written as

$$z(t) = C_0\, e^{-\gamma\omega_n t}\cos\left(\omega_n\sqrt{1-\gamma^2}\,t - \psi\right) \tag{11}$$

Equations (10) and (11) are printed on the formula sheet provided for the examination. Once you establish that the problem involves low damping (i.e., that $\gamma < 1$), you can use either equation to give the general response. As with the previous cases, you then need to find the two constants for the particular initial conditions specified in the problem.

Note that equations (10) and (11) both show that the frequency of vibration is $\omega_n\sqrt{1-\gamma^2}$. This is known as the "***damped natural frequency***" and is less than the undamped natural frequency, ω_n.

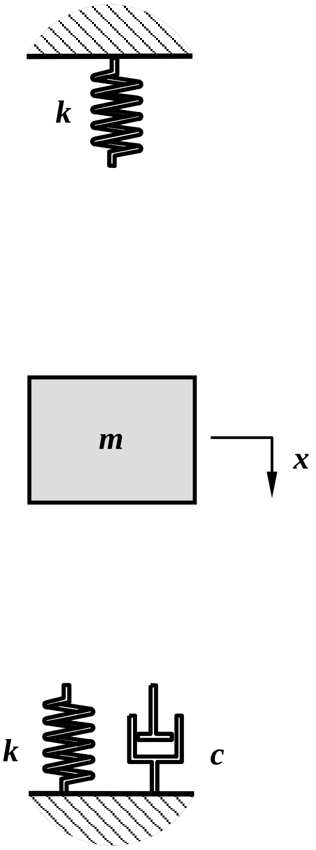
Worked Example



When at rest in equilibrium, the mass receives an impulse, $\boldsymbol{J}$, of 5 Ns applied at time, t = 0.

Find the response for t > 0.

Data: k = 500 N/m c = 20 Ns/m m = 10 kg



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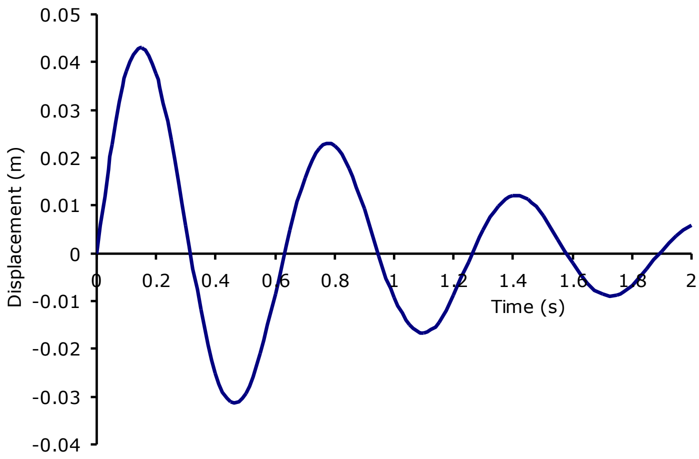
Estimating Damping

While the mass and stiffness of a structure can normally be calculated, the structural damping is very difficult to predict. However, equations (10) and (11) show that the rate of decay of the free vibration of a structure depends directly on the damping ratio and this gives a method of measuring damping.

In the above worked example, suppose we didn't know the damping value, but had done an experiment to measure the transient displacement from the impulse. Here is the measured response waveform.



We know that the expression for the displacement is $x(t) = \dfrac{J}{m\,\Omega_n}\, e^{-\gamma\,\omega_n t} \sin \Omega_n t$

The amplitude of the first peak at $t = t_1$ is $X_1 = \dfrac{J}{m\,\Omega_n}\, e^{-\gamma\,\omega_n\, t_1}$

The second peak occurs one period later and has an amplitude $X_2 = \dfrac{J}{m\,\Omega_n}\, e^{-\gamma\,\omega_n\,(t_1 + T_n)}$

The ratio of the peaks is $\dfrac{X_1}{X_2} = e^{\gamma\,\omega_n\, T_n}$

The period of the damped oscillation is $T_n = \dfrac{2\pi}{\omega_n \sqrt{1 - \gamma^2}} \approx \dfrac{2\pi}{\omega_n}$ for light damping.

Taking logs of both sides of the expression for the ratio of the peaks, we get:

$$\ln\left(\frac{X_1}{X_2}\right) = 2\,\pi\,\gamma$$

In this case, $X_1 = 0.0431\,\mathrm{m}$ and $X_2 = 0.0229\,\mathrm{m}$, so that $\gamma = 0.101$ and $c = 20.1\ \mathrm{Ns/m}$.

Note that the ratio of **any** two successive peaks is a constant, so there is good scope for making several estimates of the damping ratio from a measured response waveform.